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A NOTE ON FIXED POINTS IN SEMIMODULAR LATTICES*

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We prove that if L is a semimodular lattice of finite length (with least element 0 and greatest element 1) then the partially ordered set $L\setminus\{0, 1\}$ has the fixed point property if and only if L is not complemented. Moreover, for general lattices L of finite length we consider the relationship of the fixed point property for $L\setminus\{0, 1\}$ to several other order-theoretic conditions.

A partially ordered set P is said to have the *fixed point property* if every order-preserving map f of P to itself has a fixed point, that is, $f(a)=a$ for some $a\in P$. While the general problem of characterizing the partially ordered sets with the fixed point property remains unsolved a number of sufficient conditions as well as some necessary conditions, for this property to hold, are known (cf. [1] and [4]).

If L is a lattice with least element 0 and greatest element 1, we let $\bar{L}$ denote the partially ordered set $L-\{0, 1\}$ with the inherited partial order. We will call a partially ordered set of the form $\bar{L}$ a *reduced lattice*. The main purpose of this note is to establish the following result, which characterizes those reduced semimodular lattices of finite length which have the fixed point property. While some of the conditions below are already known to be equivalent we include them all for completeness.

Theorem. *Let L be a semimodular lattice of finite length. Then the following conditions are equivalent:*

(a) *$\bar{L}$ has the fixed point property,*
(b) *L is not complemented,*
(c) $\sup\{a \mid a$ *is an atom of* $L\}<1$,
(d) *$\bar{L}$ is dismantlable,*
(e) *$\overline{2^n}$ is not a retract of $\bar{L}$, where n is the length of L,*
(f) *$\bar{L}$ is contractible,*
(g) *$\bar{L}$ is $\mathbb{Q}$-acyclic.*

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A few words concerning terminology are in order. For the definitions of standard concepts in the theory of partially ordered sets we refer to [2]. Let P and Q be partially ordered sets. Then Q is a *retract* of P if there are order-preserving maps f of Q to P and g of P to Q such that $g \circ f$ is the identity map of Q (cf. [4]). As is customary we use 2^n to denote the Boolean lattice of all subsets of an n-element set. A partially ordered set P of finite length is *dismantlable* if the identity map of P is connected to some constant map in the cardinal power P^P (cf. [1]). The family of all finite chains of P forms a simplicial complex $\Delta(P)$. We say that P is *$\mathbb{Q}$-acyclic* if all the reduced simplicial homology groups of $\Delta(P)$ with rational coefficients, vanish, and that P is *contractible* if the geometric realization of $\Delta(P)$ is contractible as a topological space (cf. [1] and [3]).

The study of lattice automorphisms provides one reason for interest in fixed point results for reduced lattices. Let us say that a partially ordered set P has the *weak fixed point property* if every automorphism of P has a fixed point (see Example 1 below). It is known that a finite partially ordered set P has the fixed point property if and only if every retract of P has the weak fixed point property [4, Proposition 1]. Trivially, any lattice L with 0 and 1 has the weak fixed point property. What is of interest, however, is whether every automorphism of L has a *nontrivial* fixed point, that is, a fixed point different from 0 and 1. This is the case if and only if $\bar{L}$ has the weak fixed point property. For example, suppose that L is the lattice of subalgebras of a universal algebra A; for instance, A may be a group. If we know that $\bar{L}$ has the fixed point property, then it follows that every automorphism of A leaves some nontrivial subalgebra of A invariant.

Proof of the theorem. For *any* lattice L of finite length the following implications hold:

$$(*) \qquad \begin{tikzcd} (c) \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & (d) \arrow[r, Rightarrow] \arrow[dd, Rightarrow] & (a) \arrow[dr, Rightarrow] & \\ & & & (e) \\ (b) \arrow[r, Rightarrow] & (f) \arrow[r, Rightarrow] & (g) \arrow[ur, Rightarrow] & \end{tikzcd}$$

For (c)$\Rightarrow$(d) see the proof in [1, Corollary 4.3]. (d)$\Rightarrow$(a) is [1, Theorem 4.2]. Any retract of a partially ordered set with the fixed point property must itself have the fixed point property (cf. [4]), so (a)$\Rightarrow$(e). (c)$\Rightarrow$(b) is trivial. (d)$\Rightarrow$(f) is pointed out in [1, Section 4]. For (b)$\Rightarrow$(f) see [3, Theorem 3.3]. (f)$\Rightarrow$(g) is a standard result in algebraic topology.

Lemma. *Let P and Q be partially ordered sets and let Q be a retract of P. If P is ($\mathbb{Q}$-) acyclic, then Q is ($\mathbb{Q}$-) acyclic.*

Proof. Let $f: Q \to P$ and $g: P \to Q$ be order-preserving maps such that $g \circ f = \mathrm{id}_Q$. Since reduced homology is functorial on the category of partially ordered sets and order-preserving maps it follows that the group homomorphisms $f_n : \tilde{H}_n(Q) \to$

$\tilde{H}_n(P)$ and $g_n : \tilde{H}_n(P) \to \tilde{H}_n(Q)$ satisfy $g_n \circ f_n = (g \circ f)_n = (\mathrm{id}_Q)_n = \mathrm{id}_{\tilde{H}_n(Q)}$. Hence, $\tilde{H}_n(P) = 0$ implies that $\tilde{H}_n(Q) = 0$. $\square$

It is an immediate consequence that (g)$\Rightarrow$(e), since $\overline{2^n}$ is not $\mathbb{Q}$-acyclic.

We turn now to the proof that (e)$\Rightarrow$(c). To this end let us suppose that $\sup\{a \mid a \text{ is an atom of } L\} = 1$ in a semimodular lattice L of finite length n, where $n \geqslant$ [illegible] We can then select an independent set $A = \{a_1, a_2, \ldots, a_n\}$ of cardinality n, consisting of atoms of L. We know from [2, Theorem 5, p. 87] that $S = \{\sup B \mid B \subseteq A\}$ is a sublattice of L, that $S \cong 2^n$, and that for each $a, b \in S$, a covers b in L whenever a covers b in S. We shall prove that $\bar{S}$ is a retract of $\bar{L}$. (The idea of the proof in a sense runs dual to that of [1, Theorem 3.4].)

For convenience we label the n *coatoms* of S (elements covered by 1) $c_1, c_2, \ldots, c_n$ so that $a_i \not\leqslant c_i$ for each $i = 1, 2, \ldots, n$. We define a map $g : L \to S$ according to the following prescription: if a is an atom of L, then

$$g(a) = a_m,$$

where

$$m = \min\{i \mid a \not\leqslant c_i\}$$

and, in general if $x \in L$,

$$g(x) = \sup\{g(a) \mid a \text{ is an atom of } L \text{ and } a \leqslant x\}.$$

Then g is well-defined (in particular, $g(0) = 0$) and it is an easy matter to verify that g is order-preserving. Note that if a is an atom of L and $a \leqslant c_i$, then $g(a) \leqslant c_i$.

If $a \in A$, then $g(a) = a$. In general, if $x \in S$ and a is an atom of L satisfying $a \leqslant x$ then, since $a \leqslant c_i$ whenever $x \leqslant c_i$, it follows that

$$g(a) \leqslant \inf\{c_i \mid x \leqslant c_i\} = x.$$

This shows that $g(x) = x$ for each $x \in S$, whence, S is a retract of L (just take f to be the inclusion map of S into L).

We claim that $\bar{S}$ is a retract of $\bar{L}$, given by the retraction map $g \mid \bar{L}$. We need only show that $1 \notin g(\bar{L})$.

Suppose, on the contrary, that $g(x) = 1$ for some $x \in \bar{L}$. This means that $\{g(a) \mid a \leqslant x\} = A$. Now, observe that if $g(a) = a_n$ for some atom a of L, then $a \leqslant c_i$ for each $i = 1, 2, \ldots, n-1$, that is,

$$a \leqslant \inf\{c_i \mid i = 1, 2, \ldots, n-1\} = a_n \quad \text{or} \quad a = a_n.$$

In particular, $a_n \leqslant x$. Suppose that $a_j \leqslant x$ for each $j = i+1, i+2, \ldots, n$, and $a_i \not\leqslant x$. As $g(x) = 1$, $i \geqslant 2$. Now, let a be an atom of L contained in x with $g(a) = a_i$. Then $a \leqslant c_k$ for each $k = 1, 2, \ldots, i-1$, so

$$a \leqslant \inf\{c_k \mid k = 1, 2, \ldots, i-1\} = v.$$

On the other hand,

$$u = \sup\{a_j \mid j = i+1, i+2, \ldots, n\} \leqslant x \quad \text{and} \quad u \leqslant v;$$

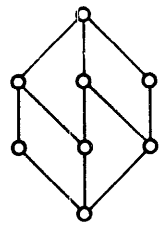


Fig. 1.

in fact, v covers u. Since $a_i \not\leq x$ it follows that $v \not\leq x$ and we conclude that $u = x \wedge v$. However, this implies that $a \leq u$, whence, $a_i = g(a) \leq g(u) = u \leq x$ which is a contradiction. $\square$

We mentioned above that the implications presented in the diagram (*) are valid for any lattice L of finite length. If L is finite it is also known that (g)⇒(a) [1, Therefore 2.1]. The list of counterexamples below shows that the implications of (*), supplemented by (g)⇒(a) and, naturally, all their composites, are the *only* implications among conditions (a)–(g) which are valid for an arbitrary finite lattice L. (Thus, in particular, the question raised in [4] whether or not a finite reduced lattice is dismantlable if it has the fixed point property has a negative answer (cf. also Example 2, below).)

(1) The lattice of Fig. 1 satisfies condition (d) but not (b).

(2) The lattice of Example 2 below satisfies condition (b) but not (d).

(3) The lattice of Example 2.4 in [1] satisfies condition (a) but not (g).

(4) Let P be the set of faces of a triangulation of the real projective plane partially ordered by set inclusion. Then P is a reduced lattice which satisfies condition (g) but not (f).

(5) The lattice of faces of a square (Fig. 2) satisfies condition (e) but not (a).

For an arbitrary lattice L of finite length we claim, that with one possible exception, the implications of the diagram (*) and their composites are the *only* valid implications among conditions (a)–(g). The exception is (b)⇒(a), to which no counterexample is presently known. In fact, it has been conjectured [3] that (b)⇒(a) is true for all lattices of finite length.

To prove our claim, we must supplement the list above with only one further counterexample.

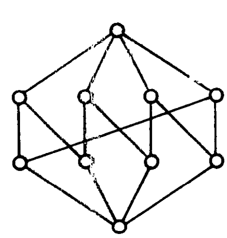


Fig. 2.

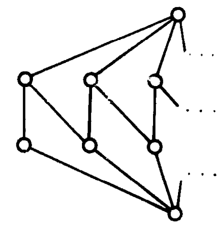


Fig. 3.

(6) Let L be the lattice of Fig. 3; that is, $\bar{L}$ is an infinite "zig-zag line". Then L satisfies condition (f) but not (a).

Example 1. Let π_n be the lattice of partitions of an n-element set, $n \geqslant 3$ (cf. [2]). π_n is semimodular and complemented. Hence $\overline{\pi_n}$ does not have the fixed point property. However, $\overline{\pi_n}$ has the weak fixed point property if and only if n is not prime. To show this we use the well-known fact that every automorphism of π_n, and hence of $\overline{\pi_n}$, is induced in the natural way by a permutation of the underlying n-element set (cf. [2, p. 81]). Suppose that n is composite, say $n = p \cdot q$, $p > 1$, $q > 1$. If the underlying permutation has only one cycle $(x_1\, x_2 \cdots x_n)$ then the partition with blocks

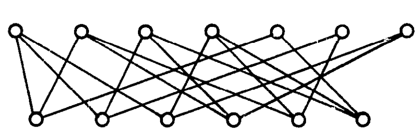
$$\{x_1, x_{p+1}, \ldots, x_{(q-1)p+1}\}, \{x_2, x_{p+2}, \ldots, x_{(q-1)p+2}\}, \ldots, \{x_p, x_{2p}, \ldots, x_n\}$$

is a fixed point of the induced automorphism. If the permutation has a cycle $(y_1 y_2 \cdots y_k)$ of length $k < n$, then the partition with blocks $\{y_1, y_2, \ldots, y_k\}$ and $\{y_{k+1}, y_{k+2}, \ldots, y_n\}$ is a fixed point. Suppose on the other hand that n is prime. Let $(x_1 x_2 \cdots x_n)$ be a one-cycle permutation of the underlying point set. It is straightforward to verify that it will induce a fixed point free automorphism of $\overline{\pi_n}$.

Example 2. We now present a construction, devised by I. Rival, of a finite noncomplemented lattice L with the property that each of its elements is the supremum of atoms and the infimum of coatoms. Using a reformulation of dismantlability, which is available in the finite case (see e.g. [1, Theorem 4.1]) it is easily seen that $\bar{L}$ cannot be dismantlable.

Let L be the set of all infima of

$$C = \{\{1,2,3,4\}, \{1,2,3,6\}, \{1,2,4,5\}, \{1,2,5,6\},$$
$$\{1,3,4,5\}, \{1,3,5,6\}, \{2,3,4,6\}, \{2,4,5,6\}\}$$



$\overline{\pi}_4$

Fig. 4.

in the lattice of all subsets of $\{1,2,3,4,5,6\}$. If we endow L with the inherited partial order then, since L is clearly a meet semilattice and has a greatest element, L is itself a lattice (the diagram of $\bar{L}$ is illustrated in Fig. 5). The set of coatoms of L is C, and by construction every element of L is the infimum of coatoms. The set of atoms of L is the set of all singletons $\{i\}$, $i=1,2,\ldots,6$, and since the partial order of L is set inclusion it follows that every element of L is the supremum of atoms. Finally, it is clear upon inspection that the element $\{1,2\}$ of L lacks a complement.

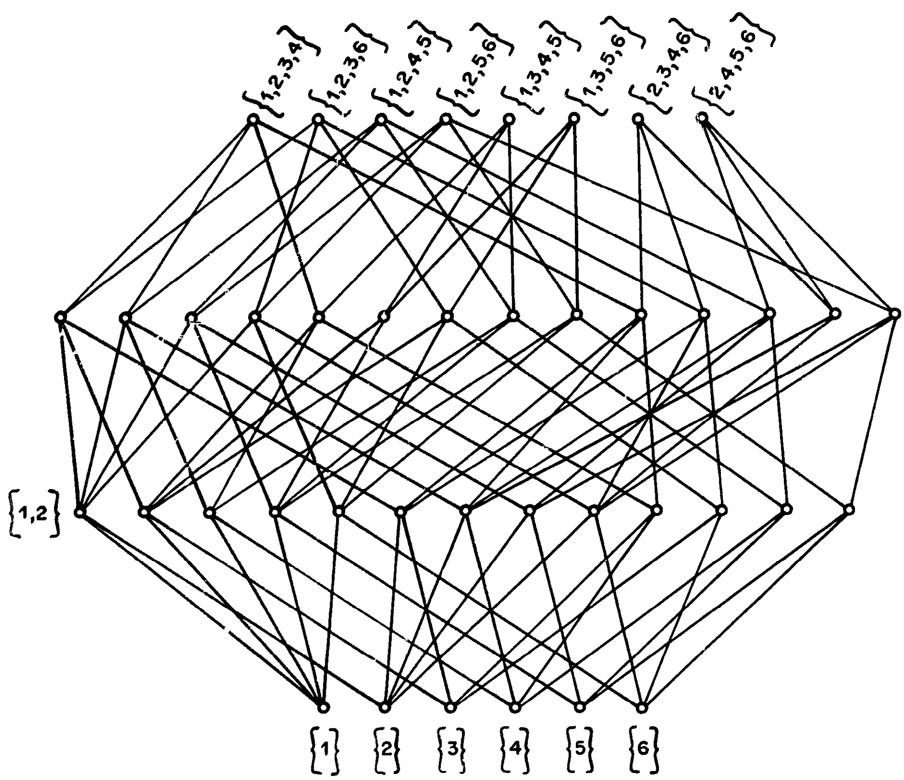


Fig. 5.

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